COSC 341 – Tutorial 11 (Solution)

1. Find regular expressions for following languages:

   (a) $L = \{a^n b^m c^l | n, m, l \in \mathbb{N}\}$ over $\Sigma = \{a, b, c\}$.

   $$a^* b^* c^*$$

   (b) $L = \{a^n b^m c^l | n, m, l \in \mathbb{N}\} \setminus \{\lambda\}$ over $\Sigma = \{a, b, c\}$.

   $$(a^+ b^* c^*) \cup (a^* b^+ c^*) \cup (a^* b^* c^+) \text{ where } a^+ = aa^*$$

   (c) $L = \{w | w \text{ contains } aa \text{ and } bb \text{ as substring}\}$ over $\Sigma = \{a, b\}$

   $$(\Sigma^* aa \Sigma^* bb \Sigma^*) \cup (\Sigma^* bb \Sigma^* aa \Sigma^*)$$

   (d) $L = \{w | w \text{ starts with } a\text{, contains two } b\text{'s and ends with } cc\}$ over $\Sigma = \{a, b, c\}$

   $$a(a \cup c)^* b(a \cup c)^* b(a \cup c)^* cc.$$

2. Is $L = \{a^n b^n c^m | m \geq n\}$ context free? Prove your answer.

   L is not context free. We prove this by using the Pumping Lemma for context free languages. For contradiction we assume that L is context free.

   We consider $z = a^k b^k c^k \in L, |z| \geq k$. Because of the Pumping Lemma there are u, v, w, x and y such that $z = uvwxy$ with $|vwx| \leq k$, $|vx| > 0$, and $uv^i wx^i y \in L$ for all $i \geq 0$. There are five possibilities of how the substring vwx could look like:

   (a) $vwx = a^j$ for some $0 < j \leq k$
   $\Rightarrow z' = uv^2 wx^2 y \in L$ according to Pumping Lemma.
   Contradiction, because z' contains more a's than c's.

   (b) $vwx = a^{j_1} b^{j_2}$ for some $0 < j_1 + j_2 \leq k$
   $\Rightarrow z' = uv^2 wx^2 y \in L$ according to Pumping Lemma.
   Contradiction, because z' contains more a's or b's than c's.

   (c) $vwx = b^j$ for some $0 < j \leq k$
   $\Rightarrow z' = uv^2 wx^2 y \in L$ according to Pumping Lemma.
   Contradiction, because z' contains more b's than c's.

   (d) $vwx = b^{j_1} c^{j_2}$ for some $0 < j_1 + j_2 \leq k$
   $\Rightarrow z' = uv^0 wx^0 y \in L$ according to Pumping Lemma.
   Contradiction, because z' contains more b's or c's than a's.

   (e) $vwx = c^j$ for some $0 < j \leq k$
   $\Rightarrow z' = uv^0 wx^0 y \in L$ according to Pumping Lemma.
   Contradiction, because z' contains more a's than c's.

   In each of the cases above we end up in a contradiction. Therefore, the assumption that L is context free was wrong. We conclude that L is not context free.

3. In each of the following cases, give examples of languages L_1 and L_2 over $\{a, b\}$ such that:

   (a) L_1 is regular, L_2 is not, and $L_1 \cup L_2$ is regular.
   $L_1 = \Sigma^*$, L_2 any non-regular language.

   (b) L_1 is regular, L_2 is not, and $L_1 \cup L_2$ is not regular.
   $L_1 = \mathsf{a}^*$, $L_2 = \mathsf{Even-Palindrome}$
   ($\mathsf{Even-Palindrome}$ is the set of strings over $\{a, b\}$ of even length that are the same spelled forward or backward)

(c) L_1 is regular, L_2 is not, and $L_1 \cap L_2$ is regular.
$L_1 = \mathbf{a}^*$, $L_2 = \mathsf{Even-Palindrome}$.

(d) L_1 is not regular, L_2 is not regular, and $L_1 \cup L_2$ is regular.
$L_1 = \mathsf{Even-Palindrome}$, $L_2 = \Sigma^* \setminus \mathsf{Even-Palindrome}$ (which cannot be regular, because the regular languages are closed under complementation.

(e) L_1 is not regular and L_1^* is regular.
$L_1 = \mathsf{Even-Palindrome} \cup \{a, b\}$.